Symmetry realization of texture zeros

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Abstract. We show that it is possible to enforce texture zeros in arbitrary entries of the fermion mass matrices by means of Abelian symmetries; in this way, many popular mass-matrix textures find a symmetry justification. We propose two alternative methods which allow one to place zeros in any number of elements of the mass matrices that one wants. They are applicable simultaneously in the quark and lepton sectors. They are also applicable in grand unified theories. The number of scalar fields required by our methods may be large; still, in many interesting cases this number can be reduced considerably. The larger the desired number of texture zeros is, the simpler are the models which reproduce the texture.

1 Introduction

The data from recent neutrino experiments [1–4] confirm the neutrino oscillation hypothesis [5] and, therefore, confirm the hypothesis [6] of neutrino masses and lepton mixing – for recent reviews see [7]. While experiments have made great progress in determining the lepton mixing matrix and the two neutrino mass-squared differences, the origin of neutrino masses and lepton mixing is still far from clear, as witnessed by the many ideas put forward in this context – see, for instance, [8]. Recently, Abelian symmetries have been studied systematically in [9] with the aim of achieving extremal mixing angles. A standard attempt at explaining the observed masses and mixing angles is provided by "textures" for the lepton mass matrices, with "texture zeros", which have been widely discussed in the literature [10–14]. In particular, it has been found that, in the weak basis where the charged-lepton mass matrix is diagonal, neutrino mass matrices with two texture zeros[1] are consistent with all the experimental data [15]. The neutrino mass matrix does not seem to display the hierarchical structure apparent in quark mass matrices [16–20].

The important theoretical problem with texture zeros is their origin. In this paper, we show that it is possible to enforce texture zeros in arbitrary entries of the fermion mass matrices by means of an Abelian symmetry. We suggest two alternative general possibilities for the Abelian symmetry group,[2] which are simultaneously applicable in both the quark and lepton sectors, and also in grand unified theories. They allow one to embed mass-matrix textures with zeros into renormalizable field theories, at the expense, sometimes, of a proliferation of scalars and flavour-changing neutral Yukawa interactions.

We explain the general methods in Sect. 2 and illustrate them with four examples in Sect. 3. Three examples are taken from the lepton sector and use the seesaw mechanism [23] to suppress the neutrino masses; the last example is from the quark sector. It is also shown in Sect. 3 that, in physically interesting cases, the number of scalars needed to implement a texture can be reduced considerably relative to the general methods of Sect. 2; unfortunately, it is not easy to provide a general way for reducing the number of scalar multiplets, or to find out the minimal number of scalars needed in order to implement a given texture. How to avoid potential Goldstone bosons in our symmetry realization of texture zeros is the subject theme of Sect. 4. The conclusions are presented in Sect. 5.

2 The methods

We explain our methods by considering the lepton sector of the standard model (SM), which has three right-handed charged-lepton singlets ℓ_{Ra} and three left-handed lepton

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[1] Since we are dealing with Majorana neutrinos, the neutrino mass matrix $\mathcal{M}_\nu$ is symmetric and $(\mathcal{M}_\nu)_{ab} = (\mathcal{M}_\nu)_{ba} = 0$ is counted as only one texture zero when $a \neq b$.

[2] In the discovery of the first alternative we have been inspired by the "dimensional deconstruction" [21] model of [22].

doublets $D_{\mathrm{L}a}$ ($a = 1, 2, 3$); to this we add three right-handed neutrino singlets $\nu_{\mathrm{R}a}$, in order to enable the seesaw mechanism for suppressing the neutrino masses.

2.1 First method

We assume that each of the nine fermion multiplets f transforms under a *separate* Abelian group $\mathcal{G}(f)$, so that the full horizontal symmetry group $\mathcal{G}$ is the direct product of nine groups:

$$\mathcal{G} = \times_f \mathcal{G}(f), \quad \text{with } f = \ell_{\mathrm{R}a}, D_{\mathrm{L}a}, \nu_{\mathrm{R}a} \ (a = 1, 2, 3). \quad (1)$$

For mathematical transparency we identify the groups $\mathcal{G}(f)$ with their unitary representations; thus, the $\mathcal{G}(f)$ are groups of complex numbers on the unit circle, either the discrete groups Z_n or the continuous group $U(1)$. We introduce Higgs doublets ϕ_{ab} and $\tilde{\phi}_{ab}$ (note that in general the ϕ_{ab} and the $\tilde{\phi}_{ab}$ are *independent* degrees of freedom), with weak hypercharges $+1/2$ and $-1/2$, respectively, together with scalar singlets χ_{ab} with weak hypercharge 0. They transform in the following way:

$$\begin{aligned} \phi_{ab} &: \mathcal{G}^*(\ell_{\mathrm{R}a}) \otimes \mathcal{G}(D_{\mathrm{L}b}), \\ \tilde{\phi}_{ab} &: \mathcal{G}^*(\nu_{\mathrm{R}a}) \otimes \mathcal{G}(D_{\mathrm{L}b}), \\ \chi_{ab} &: \mathcal{G}(\nu_{\mathrm{R}a}) \otimes \mathcal{G}(\nu_{\mathrm{R}b}). \end{aligned} \quad (2)$$

We make the identification $\chi_{ab} \equiv \chi_{ba}$, $\forall a, b \in \{1, 2, 3\}$, so that there are only six (in general complex) scalar singlets. The $\mathcal{G}$-invariant Yukawa Lagrangian of the leptons is

$$\begin{aligned} \mathcal{L}_{\mathrm{Y}\ell} = -\sum_{a,b} \Bigg(&\Gamma_{ab}\, \bar{\ell}_{\mathrm{R}a} \phi_{ab}^\dagger D_{\mathrm{L}b} + \Delta_{ab}\, \bar{\nu}_{\mathrm{R}a} \tilde{\phi}_{ab}^\dagger D_{\mathrm{L}b} \\ &+ \frac{1}{2}\, Y_{ab}\, \chi_{ab} \bar{\nu}_{\mathrm{R}a} C \bar{\nu}_{\mathrm{R}b}^{\mathrm{T}} \Bigg) + \mathrm{H.c.} \end{aligned} \quad (3)$$

There is one coupling constant Γ_{ab} for each Higgs doublet ϕ_{ab}, one coupling constant Δ_{ab} for each Higgs doublet $\tilde{\phi}_{ab}$, and one coupling constant Y_{ab} for each scalar singlet χ_{ab}. With the vacuum expectation values (VEVs)

$$\langle 0|\phi_{ab}|0\rangle = \begin{pmatrix} 0 \\ v_{ab} \end{pmatrix}, \quad \langle 0|\tilde{\phi}_{ab}|0\rangle = \begin{pmatrix} w_{ab}^* \\ 0 \end{pmatrix},$$

$$\langle 0|\chi_{ab}|0\rangle = X_{ab}, \quad (4)$$

one obtains the charged-lepton mass matrix M_ℓ, the neutrino Dirac mass matrix M_{D}, and the right-handed neutrino Majorana mass matrix M_{R} through

$$(M_\ell)_{ab} = v_{ab}^* \Gamma_{ab}, \quad (M_{\mathrm{D}})_{ab} = w_{ab} \Delta_{ab},$$

and

$$(M_{\mathrm{R}})_{ab} = X_{ab} Y_{ab}, \quad (5)$$

respectively. It is natural to assume that the VEVs X_{ab} of the scalar singlets are of a very large (seesaw) scale. One then obtains a suppressed Majorana mass matrix [23]

$$\mathcal{M}_\nu = -M_{\mathrm{D}}^{\mathrm{T}} M_{\mathrm{R}}^{-1} M_{\mathrm{D}} \quad (6)$$

for the light neutrinos.

In this scheme, according to (2)–(5), for each non-zero entry in M_ℓ or M_{D} we need one Higgs doublet, with appropriate transformation properties under $\mathcal{G}$, connecting the two fermion multiplets corresponding to that entry. Similarly, for each non-zero matrix element of M_{R} we need a singlet χ_{ab} with appropriate transformation properties under $\mathcal{G}$. Under these conditions, it is easy to place a texture zero in any entry of the mass matrices M_ℓ, M_{D}, or M_{R}, simply by not introducing into the theory the corresponding scalar multiplet. Thus, if we want to have $(M_\ell)_{ab} \neq 0$, then we allow the presence of ϕ_{ab} in the theory; if, on the contrary, we want $(M_\ell)_{ab}$ to vanish, then we do not admit the Higgs doublet ϕ_{ab} into the set of scalar fields.

For each of the matrices M_ℓ and M_{D} we need at most nine Higgs doublets; in the case of nine Higgs doublets we obtain the most general mass matrix. For a fully general M_{R}, which is symmetric, we need six scalar singlets. In practice, however, in order to obtain predictive mass matrices there will be many zero entries in those matrices, and the actual number of scalar multiplets will be much less than $9 + 9 + 6 = 24$.

If the horizontal symmetry group corresponding to $\nu_{\mathrm{R}a}$ is $\mathrm{Z}_2(\nu_{\mathrm{R}a})$, then we obtain a non-zero diagonal entry $(M_{\mathrm{R}})_{aa}$ in M_{R} from the automatic presence in the Lagrangian of the mass term $-m_a \bar{\nu}_{\mathrm{R}a} C \bar{\nu}_{\mathrm{R}a}^{\mathrm{T}} + \mathrm{H.c.}$, without the need for a singlet χ_{aa}. Thus, if one wants to enforce $(M_{\mathrm{R}})_{aa} = 0$, then $\nu_{\mathrm{R}a}$ must not transform as $\mathrm{Z}_2(\nu_{\mathrm{R}a})$, rather $\mathcal{G}(\nu_{\mathrm{R}a})$ has to be genuinely complex.

In order to obtain a non-singular M_ℓ we need at least three non-zero elements in that matrix and, by adequately interchanging the $\ell_{\mathrm{R}a}$, those three non-zero matrix elements may be placed along the diagonal of M_ℓ. The number of Higgs doublets ϕ_{ab} needed may then be reduced by assuming $\mathcal{G}(\ell_{\mathrm{R}a}) \equiv \mathcal{G}(D_{\mathrm{L}a})$ and by using only one ϕ doublet, invariant under the horizontal symmetry group, which simultaneously generates all three diagonal matrix elements in M_ℓ. A similar trick may be used in M_{D}, and also in M_{R} if one wants to reduce the number of scalar singlets employed.

If the zeros in M_ℓ and M_{D} are at the same places, and if the Higgs doublets responsible for M_ℓ transform under the same *real* representation $\mathcal{G}$ as those responsible for M_{D}, then we may identify $\tilde{\phi}_{ab}$ with $\mathrm{i}\sigma_2 \phi_{ab}^*$; in that case we have $\mathcal{G}(\ell_{\mathrm{R}a}) \equiv \mathcal{G}(\nu_{\mathrm{R}a})$, $\mathcal{G}$ is the direct product of only six groups $\mathcal{G}(f)$, and the number of Higgs doublets needed may be reduced considerably.

All these points will become clearer after the examples in Sect. 3.

2.2 Second method

An alternative and much simpler possibility for the Abelian symmetry group is the general choice

$$\mathcal{G} = \mathrm{Z}_{12} \times \mathrm{Z}_2. \quad (7)$$

Let $\omega = \exp(\mathrm{i}\pi/6)$. Under the Z_{12} of (7),

$$\bar{\ell}_{\mathrm{R}1} \to \omega\, \bar{\ell}_{\mathrm{R}1}, \quad \bar{\nu}_{\mathrm{R}1} \to \omega\, \bar{\nu}_{\mathrm{R}1}, \quad D_{\mathrm{L}1} \to \omega\, D_{\mathrm{L}1},$$

$$\bar{\ell}_{\mathrm{R}2} \to \omega^2 \bar{\ell}_{\mathrm{R}2}, \quad \bar{\nu}_{\mathrm{R}2} \to \omega^2 \bar{\nu}_{\mathrm{R}2}, \quad D_{\mathrm{L}2} \to \omega^3 D_{\mathrm{L}2}, \qquad (8)$$

$$\bar{\ell}_{\mathrm{R}3} \to \omega^5 \bar{\ell}_{\mathrm{R}3}, \quad \bar{\nu}_{\mathrm{R}3} \to \omega^5 \bar{\nu}_{\mathrm{R}3}, \quad D_{\mathrm{L}3} \to \omega^8 D_{\mathrm{L}3}.$$

Then, the bilinears $\bar{\ell}_{\mathrm{R}a} D_{\mathrm{L}b}$ and $\bar{\nu}_{\mathrm{R}a} D_{\mathrm{L}b}$, relevant for $(M_\ell)_{ab}$ and $(M_{\mathrm{D}})_{ab}$, respectively, transform according to the matrix

$$\begin{pmatrix} \omega^2 & \omega^4 & \omega^9 \\ \omega^3 & \omega^5 & \omega^{10} \\ \omega^6 & \omega^8 & \omega \end{pmatrix}, \qquad (9)$$

while the bilinears $\bar{\nu}_{\mathrm{R}a} C \bar{\nu}_{\mathrm{R}b}^{\mathrm{T}}$, relevant for $(M_{\mathrm{R}})_{ab}$, transform according to

$$\begin{pmatrix} \omega^2 & \omega^3 & \omega^6 \\ \omega^3 & \omega^4 & \omega^7 \\ \omega^6 & \omega^7 & \omega^{10} \end{pmatrix}. \qquad (10)$$

Since all the powers of ω in the matrix of (10) are different, we may introduce into the theory singlets χ_{ab} with the appropriate transformation properties under Z_{12} in order to render non-zero only those matrix elements of M_{R} that one wants; notice that no bilinear $\bar{\nu}_{\mathrm{R}a} C \bar{\nu}_{\mathrm{R}b}^{\mathrm{T}}$ is $\mathcal{G}$-invariant, and therefore χ_{ab} is always needed in order to obtain $(M_{\mathrm{R}})_{ab} \neq 0$. Similarly, all powers of ω in the matrix of (9) are different, and therefore one needs a separate ϕ_{ab} or $\tilde{\phi}_{ab}$ in order to make each matrix element $(M_\ell)_{ab}$ or $(M_{\mathrm{D}})_{ab}$, respectively, non-zero.

The factor group Z_2 in (7) is needed in order to distinguish the ϕ_{ab} from the $\tilde{\phi}_{ab}$, so that some Higgs doublets do not simultaneously generate non-zero matrix elements in M_ℓ and in M_{D}. Under that Z_2 the $\tilde{\phi}_{ab}$ and the neutrino singlets $\nu_{\mathrm{R}a}$ change sign, while all other multiplets remain invariant.

Variations on this second method are, of course, possible. One may modify the transformation properties of the $D_{\mathrm{L}a}$ by a fixed power of ω, for instance. Or one may substitute Z_{12} by a $U(1)$ group, by trading the ω^q in (8) by $e^{\mathrm{i}q\alpha}$, with α a continuous real parameter. Most important, in the realization of many specific textures one may trade the symmetry $\mathrm{Z}_{12} \times \mathrm{Z}_2$ by a smaller symmetry; we will encounter examples with Z_8 in the following section.

3 Examples

We have borrowed the first two examples from [24], where they are called cases b_2 and a_1, respectively; they reproduce the texture A_2 of [15] for $\mathcal{M}_\nu$:

$$\mathcal{M}_\nu \sim \begin{pmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}, \qquad (11)$$

where the crosses represent non-zero matrix elements. In both examples M_ℓ is diagonal, hence we may identify the indices $1, 2, 3$ with e, μ, τ, respectively.

Example A

We have

$$M_\ell \text{ and } M_{\mathrm{D}} \text{ diagonal}, \quad M_{\mathrm{R}} \sim \begin{pmatrix} 0 & \times & \times \\ \times & 0 & 0 \\ \times & 0 & \times \end{pmatrix}. \qquad (12)$$

This is a particularly simple case since M_ℓ and M_{D} are simultaneously diagonal. Using the first method of the previous section, this allows for the identification $\mathcal{G}(\ell_{\mathrm{R}a}) \equiv \mathcal{G}(\nu_{\mathrm{R}a}) \equiv \mathcal{G}(D_{\mathrm{L}a})$, so that $\mathcal{G}$ is the direct product of only three groups. One Higgs doublet ϕ transforming trivially under $\mathcal{G}$, together with $\tilde{\phi} = \mathrm{i}\sigma_2 \phi^*$, are sufficient for generating both M_ℓ and M_{D}. Since $(M_{\mathrm{R}})_{11} = (M_{\mathrm{R}})_{22} = 0$, we take $\mathcal{G}(\nu_{\mathrm{R}1}) \equiv \mathrm{Z}_4(\nu_{\mathrm{R}1})$ and $\mathcal{G}(\nu_{\mathrm{R}2}) \equiv \mathrm{Z}_4(\nu_{\mathrm{R}2})$; for $\mathcal{G}(\nu_{\mathrm{R}3})$ we use $\mathcal{G}(\nu_{\mathrm{R}3}) \equiv \mathrm{Z}_2(\nu_{\mathrm{R}3})$ so that $(M_{\mathrm{R}})_{33}$ is non-zero even in the absence of any scalar singlets. We need two scalar singlets, transforming as

$$\begin{array}{l} \chi_{12} : \mathrm{Z}_4(\nu_{\mathrm{R}1}) \otimes \mathrm{Z}_4(\nu_{\mathrm{R}2}), \\ \chi_{13} : \mathrm{Z}_4(\nu_{\mathrm{R}1}) \otimes \mathrm{Z}_2(\nu_{\mathrm{R}3}). \end{array} \qquad (13)$$

Thus, in this example there is only the SM Higgs doublet, together with two complex scalar singlets which at low energies are invisible.

Example B

We have

$$M_\ell \text{ diagonal}, \quad M_{\mathrm{D}} \sim \begin{pmatrix} \times & 0 & \times \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix},$$

$$M_{\mathrm{R}} \sim \begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & 0 \end{pmatrix}. \qquad (14)$$

Firstly, we discuss a straightforward realization of this texture, using the first method of Sect. 2. Since M_ℓ is diagonal, we choose $\mathcal{G}(\ell_{\mathrm{R}a}) \equiv \mathcal{G}(D_{\mathrm{L}a})$, and $\mathcal{G}$ is the direct product of only six groups. In order to obtain the diagonal M_ℓ, it is then sufficient to introduce one Higgs doublet ϕ, transforming trivially under $\mathcal{G}$. We further choose $\mathcal{G}(D_{\mathrm{L}a}) = \mathrm{Z}_2(D_{\mathrm{L}a}), \forall a = 1, 2, 3$. Since M_{R} has one zero in the diagonal, we take a genuinely complex group for all the $\nu_{\mathrm{R}a}$, viz. $\mathcal{G}(\nu_{\mathrm{R}a}) = \mathrm{Z}_4(\nu_{\mathrm{R}a})$. There are four non-zero entries in M_{D}, therefore we need (besides ϕ) four more Higgs doublets, with the following transformation properties:

$$\begin{array}{l} \tilde{\phi}_{11} : \mathrm{Z}_4^*(\nu_{\mathrm{R}1}) \otimes \mathrm{Z}_2(D_{\mathrm{L}1}), \\ \tilde{\phi}_{13} : \mathrm{Z}_4^*(\nu_{\mathrm{R}1}) \otimes \mathrm{Z}_2(D_{\mathrm{L}3}), \\ \tilde{\phi}_{23} : \mathrm{Z}_4^*(\nu_{\mathrm{R}2}) \otimes \mathrm{Z}_2(D_{\mathrm{L}3}), \\ \tilde{\phi}_{32} : \mathrm{Z}_4^*(\nu_{\mathrm{R}3}) \otimes \mathrm{Z}_2(D_{\mathrm{L}2}). \end{array} \qquad (15)$$

It remains to discuss the scalar singlets:

$$\begin{array}{l} \chi_{11} : \mathrm{Z}_4(\nu_{\mathrm{R}1}) \otimes \mathrm{Z}_4(\nu_{\mathrm{R}1}), \\ \chi_{22} : \mathrm{Z}_4(\nu_{\mathrm{R}2}) \otimes \mathrm{Z}_4(\nu_{\mathrm{R}2}), \\ \chi_{13} : \mathrm{Z}_4(\nu_{\mathrm{R}1}) \otimes \mathrm{Z}_4(\nu_{\mathrm{R}3}). \end{array} \qquad (16)$$

There are five Higgs doublets and three scalar singlets in this straightforward realization of the texture.

Secondly, one may try, still using the first method of Sect. 2, to reduce the number of scalars. We stick to $\mathcal{G}(\ell_{\mathrm{R}a}) \equiv \mathcal{G}(D_{\mathrm{L}a})$ and use for $\mathcal{G}$ the direct product of five groups instead of six:

$$
\begin{aligned}
& \mathrm{Z}_2\left(D_{\mathrm{L}1}\right) \equiv \mathrm{Z}_2\left(\ell_{\mathrm{R}1}\right), \\
& \mathrm{Z}_4\left(D_{\mathrm{L}2}\right) \equiv \mathrm{Z}_4\left(\ell_{\mathrm{R}2}\right) \equiv \mathrm{Z}_4\left(\nu_{\mathrm{R}3}\right), \\
& \mathrm{Z}_2\left(D_{\mathrm{L}3}\right) \equiv \mathrm{Z}_2\left(\ell_{\mathrm{R}3}\right), \\
& \mathrm{Z}_2\left(\nu_{\mathrm{R}1}\right), \\
& \mathrm{Z}_2\left(\nu_{\mathrm{R}2}\right).
\end{aligned}
\tag{17}
$$

As before, we introduce a Higgs doublet ϕ transforming trivially under $\mathcal{G}$. Then the coupling of $\tilde{\phi} = \mathrm{i}\sigma_2\phi^*$ is responsible for $(M_{\mathrm{D}})_{32} \neq 0$. For the three remaining non-zero elements of M_{D} we need the following Higgs doublets:

$$
\begin{aligned}
& \tilde{\phi}_{11} : \mathrm{Z}_2\left(\nu_{\mathrm{R}1}\right) \otimes \mathrm{Z}_2\left(D_{\mathrm{L}1}\right), \\
& \tilde{\phi}_{13} : \mathrm{Z}_2\left(\nu_{\mathrm{R}1}\right) \otimes \mathrm{Z}_2\left(D_{\mathrm{L}3}\right), \\
& \tilde{\phi}_{23} : \mathrm{Z}_2\left(\nu_{\mathrm{R}2}\right) \otimes \mathrm{Z}_2\left(D_{\mathrm{L}3}\right).
\end{aligned}
\tag{18}
$$

Now one scalar singlet χ_{13} transforming as $\mathrm{Z}_2\left(\nu_{\mathrm{R}1}\right) \otimes \mathrm{Z}_4\left(\nu_{\mathrm{R}3}\right)$ is sufficient. Thus, in this realization of the texture of (14) we manage to have only four Higgs doublets and one scalar singlet, while $\mathcal{G}$ is the direct product of four Z_2 and one Z_4 groups.

Thirdly, we note that there are even more economic realizations of the present texture, when one uses the second method of the previous section, or a simplified version thereof. Consider for instance the following Z_8 symmetry:

$$
\begin{aligned}
& \bar{\ell}_{\mathrm{R}2} \to \zeta\bar{\ell}_{\mathrm{R}2}, \quad \bar{\nu}_{\mathrm{R}2} \to \zeta^6\bar{\nu}_{\mathrm{R}2}, \quad D_{\mathrm{L}2} \to \zeta^7 D_{\mathrm{L}2}, \\
& \bar{\ell}_{\mathrm{R}3} \to \zeta^4\bar{\ell}_{\mathrm{R}3}, \quad \bar{\nu}_{\mathrm{R}3} \to \zeta^3\bar{\nu}_{\mathrm{R}3}, \quad D_{\mathrm{L}3} \to \zeta^2 D_{\mathrm{L}3},
\end{aligned}
\tag{19}
$$

where $\zeta = \exp(\mathrm{i}\pi/4)$. We then need only two Higgs doublets ϕ and ϕ', one complex scalar singlet χ, and one real scalar singlet χ', transforming as

$$
\begin{aligned}
\phi & \to \phi, \\
\phi' & \to \zeta^6\phi', \\
\chi & \to \zeta^5\chi, \\
\chi' & \to \zeta^4\chi'.
\end{aligned}
\tag{20}
$$

The Yukawa couplings matrices of ϕ, ϕ', $\tilde{\phi} = \mathrm{i}\sigma_2\phi^*$, and $\tilde{\phi}' = \mathrm{i}\sigma_2\phi'^*$ are given by

$$
\Gamma \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Gamma' \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \times \end{pmatrix},
$$

$$
\Delta \sim \begin{pmatrix} \times & 0 & 0 \\ 0 & 0 & \times \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta' \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ 0 & \times & 0 \end{pmatrix},
\tag{21}
$$

respectively. While the non-zero entry $(M_{\mathrm{R}})_{11}$ in the mass matrix of the right-handed neutrino singlets does not need a scalar singlet, the rest of the entries in M_{R} of (14) is supplied by the Yukawa coupling matrices

$$
Y \sim \begin{pmatrix} 0 & 0 & \times \\ 0 & 0 & 0 \\ \times & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y' \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix}
\tag{22}
$$

of χ and χ', respectively.

Our second and third realizations of the texture in (14) illustrate the fact that the methods presented in Sect. 2 do not necessarily lead to the simplest Abelian symmetry which justifies each texture, nor to the minimal number of scalar multiplets or fields. For some textures it may be possible to find simpler symmetries than the ones presented in Sect. 2, and realizations of the texture which require fewer scalar multiplets.

Example C

We have

$$
M_\ell \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix},
$$

$$
M_{\mathrm{D}} \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & \times \end{pmatrix}, \quad M_{\mathrm{R}} \text{ diagonal.}
\tag{23}
$$

We have borrowed this example from [25] (see also [26]), where it is moreover assumed that the diagonal elements of M_{R} are all equal; unfortunately, the latter feature cannot be implemented with the methods proposed in the present paper. In M_ℓ and M_{D} the position of the zeros is the same; using the first method in Sect. 2, this makes it convenient to choose $\mathcal{G}(\ell_{\mathrm{R}a}) \equiv \mathcal{G}(\nu_{\mathrm{R}a})$. We set $\mathcal{G}(\nu_{\mathrm{R}a}) = \mathrm{Z}_2(\nu_{\mathrm{R}a})$, an option which automatically yields a mass matrix M_{R} with non-zero diagonal elements, without the need for any scalar singlets. We choose $\mathcal{G}$ as being the direct product of four Z_2 groups:

$$
\mathrm{Z}_2\left(D_{\mathrm{L}1}\right) \equiv \mathrm{Z}_2\left(\ell_{\mathrm{R}2}\right) \equiv \mathrm{Z}_2\left(\nu_{\mathrm{R}2}\right),
\tag{24}
$$

$$
\mathrm{Z}_2\left(D_{\mathrm{L}2}\right) \equiv \mathrm{Z}_2\left(\ell_{\mathrm{R}1}\right) \equiv \mathrm{Z}_2\left(\nu_{\mathrm{R}1}\right),
\tag{25}
$$

$$
\mathrm{Z}_2\left(\ell_{\mathrm{R}3}\right) \equiv \mathrm{Z}_2\left(\nu_{\mathrm{R}3}\right),
\tag{26}
$$

$$
\mathrm{Z}_2\left(D_{\mathrm{L}3}\right).
\tag{27}
$$

Then we need four Higgs doublets: $\phi_{12} \equiv \phi_{21}$, which is invariant under $\mathcal{G}$; ϕ_{33}, which changes sign under the Z_2 groups in (26) and (27); ϕ_{23}, which changes sign under the Z_2 groups in (24) and (27); and ϕ_{32}, which changes sign under the Z_2 groups in (25) and (26). For all these Higgs doublets we need to consider the Yukawa couplings of both the ϕ_{ab} and the $\tilde{\phi}_{ab} = \mathrm{i}\sigma_2\phi_{ab}^*$. Thus, we reproduce the texture in (23) by using a horizontal symmetry group $\mathcal{G}$ which is the direct product of four Z_2 groups, with a scalar sector consisting of four Higgs doublets.

Lastly, we consider an example from the quark sector. In that sector, the quark doublets $q_{\mathrm{L}a}$, singlets $u_{\mathrm{R}a}$ with charge $2/3$, and singlets $d_{\mathrm{R}a}$ with charge $-1/3$ correspond to the $D_{\mathrm{L}a}$, $\nu_{\mathrm{R}a}$, and $\ell_{\mathrm{R}a}$, respectively, in the lepton sector. Moreover, the third term is missing from the Yukawa Lagrangian in (3).

Example D

We have

$$M_d \sim \begin{pmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}, \quad M_u \sim \begin{pmatrix} 0 & \times & 0 \\ \times & \times & \times \\ 0 & \times & \times \end{pmatrix}. \quad (28)$$

We have taken this texture from [27,28] (see also the references therein). Non-symmetric mass matrices are considered in one of the instances in [27]; using the methods in the present paper, it is possible neither to achieve symmetric mass matrices M_u and M_d, nor Hermitian ones like those in [28]. Using the first method in Sect. 2, since we want zeros in the same positions of M_u and M_d, we take $\mathcal{G}(u_{Ra}) \equiv \mathcal{G}(d_{Ra})$. The symmetry group $\mathcal{G}$ may be chosen to be the direct product of

$$\begin{array}{l} Z_2\,(q_{L1})\,, \\ Z_2\,(d_{R1}) \equiv Z_2\,(u_{R1})\,, \\ Z_2\,(q_{L2}) \equiv Z_2\,(d_{R2}) \equiv Z_2\,(u_{R2})\,, \\ Z_2\,(q_{L3}) \equiv Z_2\,(d_{R3}) \equiv Z_2\,(u_{R3})\,. \end{array} \quad (29)$$

Then, with four Higgs doublets transforming as

$$\begin{array}{l} \phi_{12} : Z_2\,(d_{R1}) \otimes Z_2\,(q_{L2})\,, \\ \phi_{21} : Z_2\,(d_{R2}) \otimes Z_2\,(q_{L1})\,, \\ \phi_{23} = \phi_{32} : Z_2\,(d_{R2}) \otimes Z_2\,(q_{L3})\,, \\ \phi_{22} = \phi_{33} : \{1\}\,, \end{array} \quad (30)$$

where $\{1\}$ denotes invariance under $\mathcal{G}$, one reproduces the texture in (28).

If one adopts a variation of the second method in Sect. 2, one can find a realization of the present texture with only three Higgs doublets. Using once again a Z_8 symmetry, take

$$\begin{array}{lll} \bar{d}_{R1} \to \bar{d}_{R1}, & \bar{u}_{R1} \to \bar{u}_{R1}, & q_{L1} \to \zeta^3 q_{L1}, \\ \bar{d}_{R2} \to \zeta^6 \bar{d}_{R2}, & \bar{u}_{R2} \to \zeta^6 \bar{u}_{R2}, & q_{L2} \to \zeta\, q_{L2}, \\ \bar{d}_{R3} \to \zeta^3 \bar{d}_{R3}, & \bar{u}_{R3} \to \zeta^3 \bar{u}_{R3}, & q_{L3} \to \zeta^6 q_{L3}. \end{array} \quad (31)$$

One then needs only three Higgs doublets ϕ_1, ϕ_2, and ϕ_3 transforming with ζ, ζ^7, and $\zeta^4 = -1$, respectively, in order to reproduce the texture in (28). The Yukawa coupling matrices are

$$\Gamma_1 \sim \Delta_2 \sim \begin{pmatrix} 0 & \times & 0 \\ \times & 0 & 0 \\ 0 & 0 & \times \end{pmatrix}, \quad \Gamma_2 \sim \Delta_1 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Gamma_3 \sim \Delta_3 \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \times \\ 0 & \times & 0 \end{pmatrix}. \quad (32)$$

4 Goldstone bosons

The Abelian symmetry group $\mathcal{G}$ is often so restrictive that it leads to accidental $U(1)$ symmetries in the scalar potential. Those $U(1)$ symmetries may either be shared by the rest of the Lagrangian, or not. Thus, there is the danger of Goldstone or pseudo-Goldstone bosons. If a Goldstone boson is a superposition of scalar singlets, or components thereof, then it is harmless, since it couples only to the right-handed neutrino singlets [29].

In some cases there are no Goldstone bosons. For instance, if all scalars transform with Z_2 groups, like for instance in example C of Sect. 3, then the terms $\left(\phi_i^\dagger \phi_j\right)^2$ with $i \neq j$ are $\mathcal{G}$-invariant and eliminate all the Goldstone bosons.

It is always possible to avoid the Goldstone bosons by breaking $\mathcal{G}$ softly in the scalar potential through terms of dimension two. We simply add to the potential all terms of the form $\phi_i^\dagger \phi_j$ with $i \neq j$. The only surviving $U(1)$ is then associated with hypercharge, and no Goldstone boson coupling to $\bar{\ell}_{Ra} D_{Lb}$ occurs. (If desirable, one may also eliminate the Goldstone bosons in the scalar-singlet sector in an analogous way.) For instance, our $Z_{12} \times Z_2$ method and the two examples in Sect. 3 which use the group $\mathcal{G} = Z_8$ need a soft breaking of this type.

We point out that the systematic soft breaking of a symmetry through all possible dimension-2 terms is not unusual in physics; indeed, it is always assumed in supersymmetric models. We further point out that we are assuming the soft breaking of $\mathcal{G}$ to occur *exclusively* through dimension-2 terms, but not through dimension-3 terms; indeed, bare mass terms $m_{ab}\bar{\nu}_{Ra} C \bar{\nu}_{Rb}^{T}$ + H.c. might also break $\mathcal{G}$ softly, but we assume them to remain absent, even when $\mathcal{G}$ is softly broken by dimension-2 terms in the scalar potential, lest the texture of M_R be disrupted.

5 Conclusions

In this paper we have suggested flexible, general, and systematic methods for enforcing zeros in arbitrary entries of the fermion mass matrices by means of Abelian symmetries. Though we have mainly concentrated on the lepton sector, the methods, as described in Sect. 2, are also applicable in the quark sector, and in grand unified theories as well. The general methods do not, however, necessarily lead to the simplest realization of each texture; in this context, "simple" means either a small Abelian group $\mathcal{G}$, a small number of scalar multiplets (Higgs doublets and SM gauge singlets), or a scalar potential with few terms.

Using the lepton sector for definiteness, we have identified two instances where a simplification of the first method, in which the Abelian group $\mathcal{G}$ is the direct product of three groups $\mathcal{G}(D_{La})$, three $\mathcal{G}(\ell_{Ra})$, and three $\mathcal{G}(\nu_{Ra})$, is possible. They are the following.
(1) Either M_ℓ or M_D is diagonal. In this case, we identify either $\mathcal{G}(\ell_{Ra})$ or $\mathcal{G}(\nu_{Ra})$, respectively, with $\mathcal{G}(D_{La})$, for $a = 1, 2, 3$, and then $\mathcal{G}$ has six Abelian factors instead of nine. If both M_ℓ and M_D are diagonal, then we make a triple identification and we arrive at three factors in $\mathcal{G}$.
(2) The texture zeros in M_ℓ and M_D are at the same positions. In that case we identify $\mathcal{G}(\ell_{Ra})$ with $\mathcal{G}(\nu_{Ra})$ for $a = 1, 2, 3$, and use a real representation of $\mathcal{G}$ for all the Higgs doublets.

A third instance, not discussed in this paper, occurs if one has a grand unified theory; in that case the number

of matter multiplets is reduced, for instance to three in $SO(10)$, and consequently the group $\mathcal{G}$ is smaller.

There is an alternative method in which $\mathcal{G}$ is always $Z_{12} \times Z_2$. When applying that method, specific textures may allow for the use of groups Z_n with $n < 12$.

In many cases, simplifications beyond the above procedures are possible; unfortunately, it is hard to provide general rules for those simplifications. Still, it is obvious that, the larger is the number of zeros in the texture, the smaller will be the number of scalar multiplets needed.

It will often prove necessary to introduce soft breaking of the Abelian group $\mathcal{G}$ in the scalar potential, through terms of dimension two, in order to avoid dangerous Goldstone bosons. Moreover, the models constructed through our methods will usually display many flavour-changing neutral Yukawa interactions, since various Higgs doublets provide the different entries of each mass matrix. Still, as our models are well defined and protected by symmetries, these effects are calculable.

In summary, the methods described in this paper allow one to embed arbitrary fermion-mass-matrix textures with texture zeros into renormalizable field theories. This lends more credibility to this popular way of constraining the mass matrices. The embedding is not unique, indeed, there is a large degree of arbitrariness in it. Having chosen one embedding, it is possible, in principle, to calculate the radiative corrections to the relations among the masses and mixing angles which were achieved by assuming texture zeros.

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